Numeric Encodings

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Representation of Positive & Negative Integral and Real Values

- A representation for both positive and negative integral values is needed
- Objectives
  - Easy to create the negative of a value
  - Easy to perform arithmetic with both positive and negative values
  - Easy to convert to and from decimal

- A representation for real numbers is needed
- Objectives are similar

Difference Between Numbers Represented on Computers and in Mathematics

- Range
  - The scope of numbers from the smallest possible to the largest possible that can be represented
- Precision
  - The number of bits (digits) of accuracy available to approximate a real value

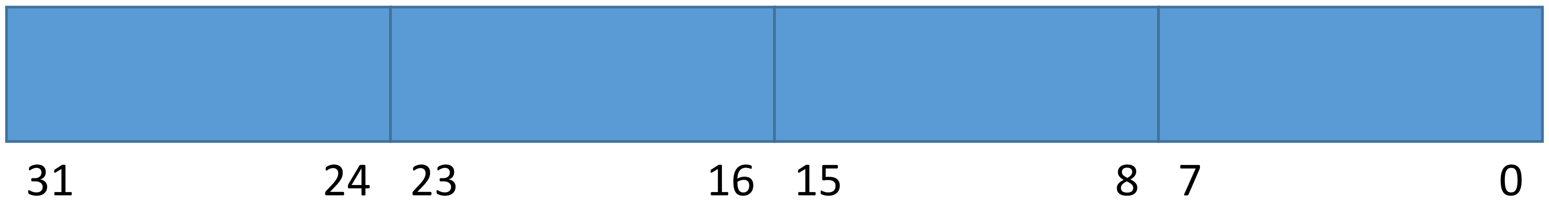
- Integral numbers in computers are limited in range
- Floating-point numbers in computers are limited in range and precision

Integral Number Representation

- Integers
  - Unsigned
  - Sign and magnitude
  - One’s-complement
  - Two’s-complement
  - Excess notation
- Range

Unsigned

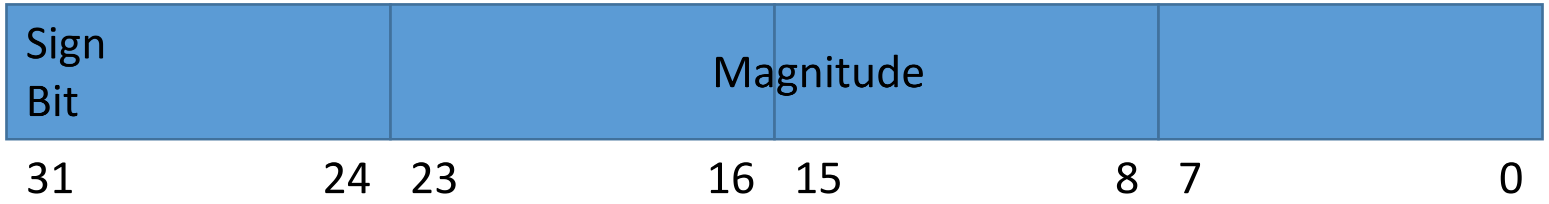
- The simplest representation allows for only positive values



- There is no way to represent negative values

Sign and Magnitude

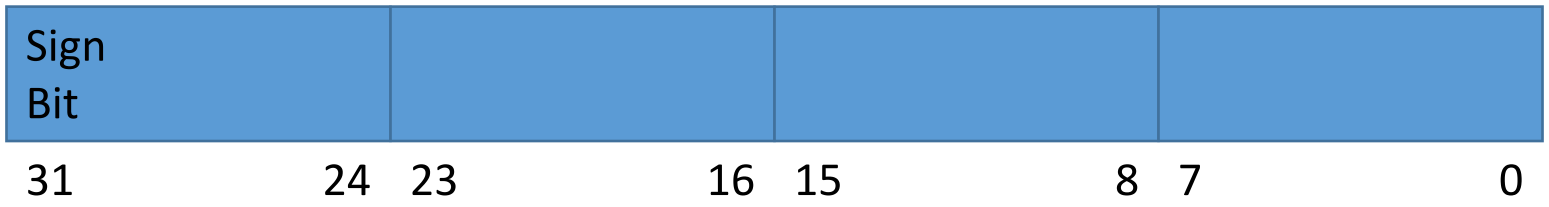
- Perhaps the next simplest representation has a sign bit followed by the value
  - Sign bit of 1 indicates a negative value
  - Sign bit of 0 indicates a positive value



- The MSB is the sign bit
  - Value = $-1^{\text{Sign-bit}} * \text{Magnitude}$
- Difficult to perform arithmetic
- Two representations for zero

One's-Complement

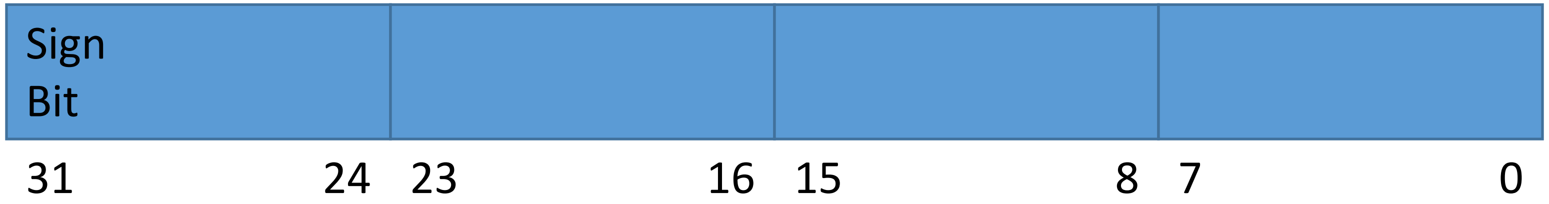
- Given a value, form its one's-complement by inverting each of the bits
- The MSB will still be used to indicate a negative value
  - Sign bit of 1 indicates a negative value
  - Sign bit of 0 indicates a positive value



- Still difficult to perform arithmetic
- Still two representations for zero

Two’s-Complement

- Given a value, form its two’s-complement by inverting each of the bits and then adding one
  - **Complement then increment**
- The MSB will still be used to indicate a negative value
  - Sign bit of 1 indicates a negative value
  - Sign bit of 0 indicates a positive value



- Easy to perform arithmetic
  - Conventional addition works with positive and negative numbers
- Only one representation for zero
- One more negative number than positive number
  - Zero has a sign bit of 0
- Two’s-complement is the most common representation for signed integral numbers

Two's-Complement and Our Adder (1 of 2)

- If we have an adder that can perform A + B + carryIn,
- Then, to perform addition, we set carryIn to 0 and the adder will add A and B

- However, if we have the option to complement either A or B and also the option to set the carryIn to 1,
- Then, using two's-complement representations, our adder will perform subtraction!
  - A + ~B + 1 = A – B
  - ~A + B + 1 = B – A

Two's-Complement and Our Adder (2 of 2)

- Furthermore, if we have the option to complement either A or B and set the other to zero and also the option to set the carryIn to 1,
- Then, using two's-complement representations, our adder will perform negation!
  - 0 + ~B + 1 = –B
  - ~A + 0 + 1 = –A

Excess Notation

- Value = Representation - Bias
- For example, using 8 bits,
  - If the representation is 64_{10} with a bias of 64_{10}, then the value is 0
  - If the representation is 65_{10} with a bias of 64_{10}, then the value is 1_{10}
  - If the representation is 63_{10} with a bias of 64_{10}, then the value is -1_{10}

- Although not easy to perform arithmetic, allows the demarcation point between positive and negative numbers to be set
- Only one representation for zero
- Used within floating-point numbers

Range of Values Represented

- Assume 8-bit word size
- 256 different bit representations

Representation	Minimum Value	Maximum Value
Unsigned	0	255
One's-complement	-127	127
Two's-complement	-128	127
Excess Notation, Bias=64_{10}	-64	191

Floating-Point Number Representation

- s sign bit (0 for positive, 1 for negative)
- b base or radix of the representation
- e exponent value (represented using excess notation with a bias)
- p number of base-b digits in the significand
- f_k significand digits
- $x = -1^s \times b^e \times (\Sigma \text{ (k=1 to p) } f_k \times b^{-k})$,
  $e_{min} \leq e \leq e_{max}$

Floating-Point Bit Configuration

- The sign bit is the MSB
- Followed by the exponent value
- The significand digits are in the LSBs

IEEE 754 Floating-Point

- Size = 32 bits (float), 64 bits (double)
- Radix = 2
- Sign bit field
- Exponent field = 8 bits (float), 11 bits (double)
- Fraction field = 23 bits (float), 52 bits (double)
- Bias = 127 (float), 1023 (double)
- Zero value representation has exponent field = 0, fraction field = 0
  - Can be positive or negative

Normalization

- A normalized number has $f_1 > 0$, if x (*i.e.,* the value) is not 0
- A subnormal (denormalized) number is non-zero, has $e = e_{min}$ and $f_1 = 0$
  - Exponent is -126 (float), -1022 (double)
- An unnormalized number is non-zero, has $e > e_{min}$ and $f_1 = 0$
- A subnormal number is too small to be normalized
- Hidden bit
  - For normalized numbers, there is an assumed single 1 bit to the left of the binary point
  - Gives one more significant bit

Special Values

- Infinities
  - Positive
  - Negative
  - *sign* = 0 for positive infinity, 1 for negative infinity; *biased exponent* = all 1 bits; *fraction* = all 0 bits
- NaN's
  - Quiet
  - Signaling
  - *sign* = either 0 or 1; *biased exponent* = all 1 bits; *fraction* = anything except all 0 bits (because all 0 bits represents infinity)

Range and Precision of Values Represented

Representation	Closest to Zero	Furthest from Zero	Precision
float	$\pm 1.18 \times 10^{-38}$	$\pm 3.4 \times 10^{38}$	~7 decimal digits
double	$\pm 2.23 \times 10^{-308}$	$\pm 1.80 \times 10^{308}$	~15 decimal digits